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Rotational compressible inviscid flow with rolled vortex sheets. An analytical algorithm for the computation of the core

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Previous work on irrotational incompressible inviscid flow (Guiraud & Zeytounian 1977) is extended to rotational and compressible flow. A formal proof is given that, within the core, one may avoid computing with the sheet by defining an equivalent continuous flow. One shows how the vorticity and the entropy gradient between the turns of the sheet are transported along trajectories of the equivalent continuous flow.

1. Introduction

The flow configuration with a rolled vortex sheet is a very basic one in fluid dynamics. It appears as one of the ways in which vorticity concentrates in slender regions. Two classical examples are vortex concentration in vortex filaments trailing behind the tips of wings (Widnall 1975) and vortex concentration in the leading-edge vortex of swept wings (Smith 1968), but many other situations are relevant, namely, for example, vortex concentration in shear flows (Damms & Kücheman 1974; Patnaik, Sherman & Corcos 1976; Corcos & Sherman 1976).

This explains the great deal of effort which has been devoted to the purpose of devising efficient numerical schemes for the computation of rolled vortex sheets (Moore 1974; Chorin & Bernard 1973; Fink & Soh 1978). Up to now, all the schemes which have been used fail to give a representation of the core when the turns of the sheet are very closely spaced.

Clearly, this belongs to the somewhat vaguely defined category of stiff problems. What is stiff in this flow configuration is the very rapid variation of various quantities transversely to the sheet, so that an accurate numerical treatment would necessitate an exceedingly refined mesh in the core region.

Clearly there is a need for a lucid combination of analytical and numerical approach as to the problem of rolled vortex sheets. Following Guiraud & Zeytounian (1977), to which we will refer as GZ in what follows, Huberson (1980) succeeded in a preliminary investigation of this nature.

The work of GZ deals with incompressible irrotational flow with embedded tightly wound rolled vortex sheets. The main result of that work may be expressed quite simply. It states that one may define an equivalent continuous, rotational, incompressible flow, ruled by Euler equations. This statement eliminates the need of computing with an exceedingly refined mesh, adapted to the small spacing between the turns of the sheet. The theory provides an algorithm which, in an asymptotic sense, is able to rebuild *a posteriori* the irrotational flow with vorticity concentrated on the sheet. Huberson has devised a numerical algorithm which realizes a shift from the computation with point vortices, in the region outside of the core, to one with spread vorticity within it. The technique was applied to the problem of rolling-up of a two-dimensional unsteady vortex sheet, with elliptical distribution of vorticity initially concentrated on a straight segment. As is well known, this configuration simulates the sheet trailing behind a wing.

Here we extend the work of GZ in order to deal with flows which are both compressible and rotational. The same idea appears to work as well. One may define an equivalent continuous flow obeying the compressible Euler equations and, from it, we are able to devise an analytical algorithm which allows us to rebuild, in an asymptotic sense, the actual flow with a rolled sheet.

The main new feature is the occurrence of entropy gradients and of two kinds of vorticity. There is first the distributed vorticity and then the one concentrated on the sheets. Both are summed up in the equivalent continuous flow, and the analytical algorithm must be able to separate them out. The same will be done with the entropy gradient.

Previous works on the analytical description of rolled vortex sheets, in an asymptotic sense, are by Mangler & Weber (1967) and by Moore (1975) on the problem dealt with in GZ. On the other hand, Brown & Mangler (1967) have extended the work of Mangler & Weber (1967) to compressible, barotropic, irrotational flow. Of course, the main new feature brought into the scheme by rotationality and entropy gradient is absent in their analysis.

2. The double-scale structure

We work with the compressible Euler equations written in dimensionless, conservative form

$$\frac{\partial \mathscr{V}}{\partial t}+\frac{\partial}{\partial x_k}(\mathscr{F}_k(\mathscr{V})) = 0, \tag{1}$$

where

$$\left.\begin{aligned} \mathscr{V} &= (\rho, \rho u_1, \rho u_2, \rho u_3, \rho S); \\ \mathscr{F}_k(\mathscr{V}) &= \left(\rho u_k, \rho u_1 u_k+\frac{p}{\gamma M_0^2}\delta_{1k}, \rho u_2 u_k+\frac{p}{\gamma M_0^2}\delta_{2k}, \rho u_3 u_k+\frac{p}{\gamma M_0^2}\delta_{3k}, \rho u_k S\right).\end{aligned}\right\} \tag{2}$$

Here, t is time; x_1, x_2, x_3 are the Cartesian components of the position vector $\mathbf{x}$; u_1, u_2, u_3 are the Cartesian components of the velocity vector $\mathbf{u}$; ρ is the density, p the pressure, S the specific entropy and γ is the ratio of specific heats. The characteristic, constant, Mach number M_0 appears as a result of working in dimensionless form. To (1) we must add an equation of state, namely

$$p = \rho^{\gamma} \exp(S). \tag{3}$$

It will be convenient to consider as the basic dependent variables p, $\mathbf{u}$ and S and we write in a shorthand notation

$$\mathscr{U} = (p, u_1, u_2, u_3, S). \tag{4}$$

It is fairly obvious that $\mathscr{V}$ is a function of $\mathscr{U}$ and that

$$d\mathscr{V} = \mathbf{A}_0(\mathscr{U})\, d\mathscr{U}, \quad d\mathscr{F}_k = \mathbf{A}_k(\mathscr{U})\, d\mathscr{U}, \tag{5}$$

where $\mathbf{A}_0$ and $\mathbf{A}_k$ are 5×5 matrices depending on $\mathscr{U}$. We shall use the vorticity

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{u}, \tag{6}$$

and the classical vorticity equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\,.\,\nabla\right)\boldsymbol{\omega} - (\boldsymbol{\omega}\,.\,\nabla)\,\mathbf{u} + (\nabla\,.\,\mathbf{u})\,\boldsymbol{\omega} = \frac{1}{\gamma M_0^2}\frac{\nabla\rho \wedge \nabla p}{\rho^2}. \tag{7}$$

We refer to GZ for a thorough discussion of the double scale concept as used in this problem and we merely state here that we look for an approximate solution in the following form

$$\mathscr{U}(t, \mathbf{x}) = \mathscr{U}^*(t, \mathbf{x}, \chi(t, \mathbf{x})), \tag{8}$$

where χ is a normalized fast variable which describes the process of crossing the turns of the rolled vortex sheet. More precisely, we require that the function χ be such that the rolled sheet be mapped onto

$$\chi = (2k+1)\,\pi, \tag{9}$$

where k runs through the positive and negative integers. Now, in the spirit of short-wave asymptotics (Whitham 1974, cha. 14; Germain 1971, 1977), we may put in a very simple mathematical way the assumption that the sheet is tightly wound; it suffices to set

$$\frac{\partial\chi}{\partial t} = C^{-1}\theta, \quad \nabla\chi = C^{-1}\mathbf{k}, \tag{10}$$

where $C \ll 1$, is a small closeness parameter.

Following GZ we normalize the short scale structure by demanding that $\mathscr{U}^*$ be 2π-periodic with respect to χ. We define an averaging operation with respect to χ in the following way

$$\bar{f} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f\, d\chi \tag{11}$$

and we write

$$f = \bar{f} + \tilde{f}, \quad \bar{\tilde{f}} = 0. \tag{12}$$

From (8), (10) the equation (1) reads

$$\theta\frac{\partial\mathscr{V}^*}{\partial\chi} + k_j\frac{\partial\mathscr{F}_j^*}{\partial\chi} + C\left(\frac{\partial\mathscr{V}^*}{\partial t} + \frac{\partial\mathscr{F}_k^*}{\partial x_k}\right) = 0, \tag{13}$$

and, in a similar way, (7) may be rewritten as follows:

$$\begin{aligned}(\theta + \mathbf{k}\,.\,\mathbf{u}^*)\frac{\partial\boldsymbol{\omega}^*}{\partial\chi} - (\mathbf{k}\,.\,\boldsymbol{\omega}^*)\frac{\partial\mathbf{u}^*}{\partial\chi} + \left(\mathbf{k}\,.\,\frac{\partial\mathbf{u}^*}{\partial\chi}\right)\boldsymbol{\omega}^* + \frac{1}{\gamma M_0^2}\frac{1}{\rho^{*2}}\left\{\frac{\partial p^*}{\partial\chi}(\mathbf{k} \wedge \nabla\rho^*) + \frac{\partial\rho^*}{\partial\chi}(\nabla p^* \wedge \mathbf{k})\right\}\\ + C\left\{\left(\frac{\partial}{\partial t} + \mathbf{u}^*\,.\,\nabla\right)\boldsymbol{\omega}^* - (\boldsymbol{\omega}^*\,.\,\nabla)\,\mathbf{u}^* + (\nabla\,.\,\mathbf{u}^*)\,\boldsymbol{\omega}^* + \frac{1}{\gamma M_0^2}\frac{\nabla p^* \wedge \nabla\rho^*}{\rho^{*2}}\right\} = 0,\end{aligned} \tag{14}$$

while, from the definition of vorticity (6), we get

$$\mathbf{k} \wedge \frac{\partial\mathbf{u}^*}{\partial\chi} + C(\nabla \wedge \mathbf{u}^* - \boldsymbol{\omega}^*) = 0. \tag{15}$$

The occurrence of the small parameter C in (13), (14) and (15) will allow us to work out an expansion technique and to derive the algorithm that is referred to in § 1. Previous to doing this we remark that, thanks to the periodicity with respect to χ, we obtain a useful result by applying the averaging operation (11) to (13). In carrying out this process we need to be cautious because $\mathscr{V}^*$ and $\mathscr{F}_j^*$ are discontinuous across the sheet. Fortunately, if we set $[f]$ for the discontinuity of f, we have

$$[\theta \mathscr{V}^* + k_j \mathscr{F}_j^*] = 0 \tag{16}$$

and this allows us to write down immediately the average of (13) in the following form

$$\frac{\partial \overline{\mathscr{V}^*}}{\partial t} + \frac{\partial \overline{\mathscr{F}_k^*}}{\partial x_k} = 0. \tag{17}$$

For further reference we state here the conditions to be satisfied on the sheet, namely the kinematic condition

$$\theta + \mathbf{k}.\mathbf{u}^* = 0, \quad \text{on} \quad \chi = (2k+1)\pi, \tag{18}$$

and the dynamic one

$$[p^*] = 0. \tag{19}$$

3. The closeness expansion process

We take care of the fact that the turns of the sheet are closely spaced by expanding with respect to C as follows:

$$\left.\begin{aligned} \mathscr{U}^* &= \mathscr{U}_0^* + C\mathscr{U}_1^* + \dots, \\ (\theta, \mathbf{k}) &= (\theta_0, \mathbf{k}_0) + C(\theta_1, \mathbf{k}_1) + \dots, \end{aligned}\right\} \tag{20}$$

and, similarly,

$$\boldsymbol{\omega}^* = \boldsymbol{\omega}_0^* + C\boldsymbol{\omega}_1^* + \dots . \tag{21}$$

To zero order from (13), (14), (15), (3) we find

$$\left.\begin{aligned} &(\theta_0 + \mathbf{k}_0.\mathbf{u}_0^*)\frac{\partial \rho_0^*}{\partial \chi} + \rho_0^* \mathbf{k}_0.\frac{\partial \mathbf{u}_0^*}{\partial \chi} = 0; \\ &\rho_0^*(\theta_0 + \mathbf{k}_0.\mathbf{u}_0^*)\frac{\partial \mathbf{u}_0^*}{\partial \chi} + \frac{1}{\gamma M_0^2}\frac{\partial p_0^*}{\partial \chi}\mathbf{k}_0 = 0; \\ &(\theta_0 + \mathbf{k}_0.\mathbf{u}_0^*)\frac{\partial S_0^*}{\partial \chi} = 0; \\ &(\theta_0 + \mathbf{k}.\mathbf{u}_0^*)\frac{\partial \boldsymbol{\omega}_0^*}{\partial \chi} - (\mathbf{k}_0.\boldsymbol{\omega}_0^*)\frac{\partial \mathbf{u}_0^*}{\partial \chi} + \left(\mathbf{k}_0.\frac{\partial \mathbf{u}_0^*}{\partial \chi}\right)\boldsymbol{\omega}_0^* \\ &\qquad + \frac{1}{\gamma M_0^2}\frac{1}{\rho_0^{*2}}\left\{\frac{\partial p_0^*}{\partial \chi}(\mathbf{k}_0 \wedge \nabla \rho_0^*) + \frac{\partial \rho_0^*}{\partial \chi}(\nabla p_0^* \wedge \mathbf{k}_0)\right\} = 0; \\ &\mathbf{k}_0 \wedge \frac{\partial \mathbf{u}_0^*}{\partial \chi} = 0; \\ &p_0^* = \rho_0^{*\gamma} \exp(S_0^*). \end{aligned}\right\} \tag{22}$$

On the other hand, from (18) and (19) we get

$$\theta_0 + \mathbf{k}_0.\mathbf{u}_0^* = 0 \quad \text{on} \quad \chi = (2k+1)\pi, \tag{23}$$

$$[p_0^*] = 0. \tag{24}$$

We shall prove in a moment that

$$\mathbf{k}_0 . \frac{\partial \mathbf{u}_0^*}{\partial \chi} = 0, \tag{25}$$

but, delaying the proof, we draw the consequences.

First, from (23) we conclude that

$$\theta_0 + \mathbf{k}_0 . \mathbf{u}_0^* = 0 \tag{26}$$

holds throughout. Then, from (22) we get

$$\frac{\partial \mathbf{u}_0^*}{\partial \chi} = 0, \quad \frac{\partial p_0^*}{\partial \chi} = 0, \quad (\nabla p_0^* \wedge \mathbf{k}_0) \frac{\partial \rho_0^*}{\partial \chi} = 0, \tag{27}$$

and observing that $\nabla p_0^* \wedge \mathbf{k}_0 \neq 0$, except at exceptional points, we obtain

$$\frac{\partial \mathbf{u}_0^*}{\partial \chi} = \frac{\partial p_0^*}{\partial \chi} = \frac{\partial \rho_0^*}{\partial \chi} = \frac{\partial S_0^*}{\partial \chi} = 0. \tag{28}$$

From (28) and (17) it is obvious that the zeroth-order quantities $\mathbf{u}_0^*, p_0^*, S_0^*$ must be a solution of the classical compressible Euler equations:

$$\frac{\partial \mathscr{V}_0^*}{\partial t} + \frac{\partial \mathscr{F}_k^*(\mathscr{V}_0^*)}{\partial x_k} = 0. \tag{29}$$

This is a very simple proof of the correspondence between the flow with the tightly wound rolled sheet and the continuous flow without a sheet. When we proceed to the next order we shall find how the sheet and the corresponding discontinuities may be recovered. For the present time we give a formal proof of (26). Assume the contrary, then $\theta_0 + \mathbf{k}_0 . \mathbf{u}_0^* \neq 0$ away from the sheet, except at isolated values of χ. From the third of (22) we find that $\partial S_0^* / \partial \chi = 0$ and, a_0^* being the non-dimensional speed of sound,

$$\frac{\partial p_0^*}{\partial \chi} = a_0^{*2} \frac{\partial \rho_0^*}{\partial \chi}. \tag{30}$$

If we eliminate $\partial \mathbf{u}_0^* / \partial \chi$ from the first two of (22) we obtain

$$(\theta_0 + \mathbf{k}_0 . \mathbf{u}_0^*)^2 \frac{\partial \rho_0^*}{\partial \chi} - \frac{|\mathbf{k}_0|^2}{\gamma M_0^2} \frac{\partial p_0^*}{\partial \chi} = 0, \tag{31}$$

and, as it is obvious that $\chi =$ const. is not a sound wave, (30) and (31) imply that $\partial p_0^* / \partial \chi = \partial \rho_0^* / \partial \chi \equiv 0$, but, then, the first of (22) is in contradiction with the assumption that (25) does not hold throughout.

4. Transport equations for the saw-tooth structure

As in GZ the saw-tooth structure of the vorticity profile transverse to the sheet is obtained when we go to the first order and try to compute $\mathbf{u}_1^*, p_1^*, \rho_1^*$ and S_1^*. After some very simple algebra we find

$$\left.\begin{aligned} |\mathbf{k}_0|^2 \frac{\partial \mathbf{u}_1^*}{\partial \chi} &= \mathbf{k}_0 \wedge (\nabla \wedge \mathbf{u}_0^*) - \mathbf{k}_0 \wedge \boldsymbol{\omega}_0^*, \\ \frac{\partial p_1^*}{\partial \chi} &= 0, \\ a_0^{*2} \frac{\partial \rho_1^*}{\partial \chi} + p_0^* \frac{\partial S_1^*}{\partial \chi} &= 0. \end{aligned}\right\} \tag{32}$$

From the first of (32) we see that $\mathbf{k}_0 . \partial \mathbf{u}_1^*/\partial \chi = 0$ and from the expansion of (18) to $O(C)$ we easily conclude that

$$\theta_1 + \mathbf{k}_0 . \mathbf{u}_1^* + \mathbf{k}_1 . \mathbf{u}_0^* = 0, \tag{33}$$

holds throughout.

Of course the equations (32) do not suffice for the determination of the dependence of $\mathbf{u}_1^*, p_1^*, \rho_1^*, S_1^*$ on χ, but we may find other equations as follows. First, if we expand the last of (1) to $O(C^2)$ it is found that

$$\frac{\partial S_1^*}{\partial t} + \mathbf{u}_0^* . \nabla S_1^* + \mathbf{u}_1^* . \nabla S_0^* = 0, \tag{34}$$

provided that (33) is taken care of. A second equation may be found by expanding (14) to $O(C)$, namely

$$-(\mathbf{k}_0 . \boldsymbol{\omega}_0^*)\frac{\partial \mathbf{u}_1^*}{\partial \chi} + \frac{1}{\gamma M_0^2}\frac{1}{\rho_0^{*2}}\frac{\partial \rho_1^*}{\partial \chi}(\nabla p_0^* \wedge \mathbf{k}_0) + \left(\frac{\partial}{\partial t} + \mathbf{u}_0^* . \nabla\right)\boldsymbol{\omega}_0^* - (\boldsymbol{\omega}_0^* . \nabla)\,\mathbf{u}_0^*$$
$$+ (\nabla . \mathbf{u}_0^*)\,\boldsymbol{\omega}_0^* - \frac{1}{\gamma M_0^2}\frac{1}{\rho_0^{*2}}(\nabla \rho_0^* \wedge \nabla p_0^*) = 0. \tag{35}$$

This equation may be modified by remarking that, as a consequence of (29), we have

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0^* . \nabla\right)(\nabla \wedge \mathbf{u}_0^*) - \{(\nabla \wedge \mathbf{u}_0^*) . \nabla\}\,\mathbf{u}_0^* + (\nabla . \mathbf{u}_0^*)(\nabla \wedge \mathbf{u}_0^*)$$
$$+ \frac{1}{\gamma M_0^2}(\nabla p_0^* \wedge \nabla \rho_0^*) = 0, \tag{36}$$

so that remembering the first of (32) we find

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0^* . \nabla\right)\left(\mathbf{k}_0 \wedge \frac{\partial \mathbf{u}_1^*}{\partial \chi}\right) - \left\{\left(\mathbf{k}_0 \wedge \frac{\partial \mathbf{u}_1^*}{\partial \chi}\right) . \nabla\right\}\mathbf{u}_0^* + (\nabla . \mathbf{u}_0^*)\left(\mathbf{k}_0 \wedge \frac{\partial \mathbf{u}_1^*}{\partial \chi}\right)$$
$$- \{\mathbf{k}_0 . (\nabla \wedge \mathbf{u}_0^*)\}\frac{\partial \mathbf{u}_1^*}{\partial \chi} + \frac{1}{\gamma M_0^2}\frac{1}{\gamma \rho_0^*}(\mathbf{k}_0 \wedge \nabla p_0^*)\frac{\partial S_1^*}{\partial \chi} = 0. \tag{37}$$

Setting

$$\frac{D_0}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}_0^* . \nabla, \tag{38}$$

let us consider the following system of two transport equations along the trajectories:

$$\frac{D_0 \mathbf{V}}{Dt} - (\mathbf{V} . \nabla)\,\mathbf{u}_0^* + (\nabla . \mathbf{u}_0^*)\,\mathbf{V} + \frac{\mathbf{k}_0 . (\nabla \wedge \mathbf{u}_0^*)}{|\mathbf{k}_0|^2}\mathbf{k}_0 \wedge \mathbf{V} + \frac{1}{\gamma M_0^2}\frac{\mathbf{k}_0 \wedge \nabla p_0^*}{\gamma \rho_0^*}\Sigma = 0, \tag{39a}$$

$$\frac{D_0 \Sigma}{Dt} + \frac{\mathbf{k}_0 \wedge \nabla S_0^*}{|\mathbf{k}_0|^2} . \mathbf{V} = 0. \tag{39b}$$

In a Lagrangian representation of the flow $\mathbf{u}_0^*$, p_0^*, ρ_0^* and S_0^* this appears to be a system of ordinary differential equations. We observe that χ does not appear in this system. Let

$$(\mathbf{V}, \Sigma) = \mathscr{T}(t, t^0)\,(\mathbf{V}^0, \Sigma^0), \tag{40}$$

be the solution of this system corresponding to initial data $\mathbf{V}^0$, Σ^0, for $t = t^0$, then we have

$$(\mathbf{k}_0 \wedge \tilde{\mathbf{u}}_1^*, \tilde{S}_1^*) = \mathscr{T}(t, t^0)\,((\mathbf{k}_0 \wedge \tilde{\mathbf{u}}_1^*)^0, \tilde{S}_1^{*0}). \tag{41}$$

This allows us to compute $\tilde{\mathbf{u}}_1^*$ and $\tilde{S}_1^*$ at any time provided they are known at some initial time t^0. We observe that the transport operator $\mathscr{T}(t, t^0)$ does not depend on χ.

We may use (41) to compute the transport of the discontinuities associated with the sheet, namely

$$(\mathbf{k}_0 \wedge [\mathbf{u}_1^*], [S_1^*]) = \mathscr{T}(t, t^0)\, ((\mathbf{k}_0 \wedge [\mathbf{u}_1^*])^0, [S_1^*]^0), \tag{42}$$

and we observe that, once these discontinuities have been transported, no new effort is necessary in order to detail the saw-tooth structure of the velocity and entropy. The corresponding signatures are transported without change, only their amplitudes being changed during the transport process.

In order that the transport process be consistent we need to verify that it preserves $\mathbf{k}_0 . \mathbf{V} = 0$. This is easily done. Taking the gradient of (26) we find a transport equation for $\mathbf{k}_0$, namely

$$\frac{D_0 \mathbf{k}_0}{Dt} + (\nabla \mathbf{u}_0^*) . \mathbf{k}_0 = 0. \tag{43}$$

We multiply (39*a*) through by $\mathbf{k}_0$ and (43) through by $\mathbf{V}$ and add the two results to get

$$\frac{D_0}{Dt}(\mathbf{k}_0 . \mathbf{V}) + (\nabla . \mathbf{u}_0^*)(\mathbf{k}_0 . \mathbf{V}) = 0, \tag{44}$$

from which it is obvious that the property $\mathbf{k}_0 . \mathbf{V} = 0$ is preserved under transport.

We complete the solution to $O(C)$ by stating that

$$\left.\begin{aligned} \mathbf{u}_1^* &= \overline{\mathbf{u}_1^*} + \tilde{\mathbf{u}}_1^*, \\ p_1^* &= \overline{p_1^*}, \\ S_1^* &= \overline{S_1^*} + \tilde{S}_1^* \\ \rho_1^* &= \overline{\rho_1^*} - \frac{p_0^*}{a_0^{*2}} \tilde{S}_1^*, \end{aligned}\right\} \tag{45}$$

and that $\overline{\mathbf{u}_1^*}, \overline{p_1^*}, \overline{S_1^*}, \overline{\rho_1^*}$ are solutions of

$$\left.\begin{aligned} \mathbf{A}_0(\mathscr{U}_0^*) \frac{\partial \overline{\mathscr{U}_1^*}}{\partial t} + \mathbf{A}_k(\mathscr{U}_0^*) \frac{\partial \overline{\mathscr{U}_1^*}}{\partial x_k} &= 0, \\ \overline{p_1^*} &= a_0^{*2} \overline{\rho_1^*} + p_0^* \overline{S_1^*}, \end{aligned}\right\} \tag{46}$$

which is the system of compressible Euler equations linearized about the solution $\mathbf{u}_0^*, p_0^*, \rho_0^*, S_0^*$. We add a comment concerning the transport equations (39) in the case considered in GZ. First the second equation does not occur, simply because there is no entropy variation to take care of. Second the first transport equation seems to disappear in GZ but it is simply trivially satisfied as a consequence of the fact that, then, $\partial \mathbf{u}_1^* / \partial \chi$ is proportional to $\nabla \wedge \mathbf{u}_0^*$ and that the condition $\mathbf{k}_0 . \nabla \wedge \mathbf{u}_0^* = 0$, holds in this case. As matter of fact it was shown in GZ that a transport equation for the vorticity concentrated on the sheet had to be satisfied, a condition which is fulfilled as a consequence of the equation corresponding to (36).

A check on the theory is obtained by deriving the Brown & Mangler (1967) solution from Brown (1965) axisymmetric compressible solution for an inviscid homentropic leading-edge vortex. This last one is obtained under the slenderness assumption. Let $\mathbf{u}_0^*$ be this solution for the velocity field; we need only check that Brown & Mangler (1967) solution may be written as

$$\mathbf{u} = \mathbf{u}_0^* + \frac{\nabla \chi_0 . (\nabla \wedge \mathbf{u}_0^*)}{|\nabla \chi_0|^2} \chi + O(\epsilon^2). \tag{47}$$

Here ϵ stands for the slenderness parameter ($r/x = O(\epsilon)$, in Brown & Mangler notations). We may check that $C = O(\epsilon^2)$ but due to the slenderness

$$[\nabla\chi_0 \cdot (\nabla \wedge \mathbf{u}_0^*)]/|\nabla\chi_0|^2$$

is $O(\epsilon^{-1})$ so that the second term is $O(\epsilon)$. As a matter of fact the Brown & Mangler (1967) solution according to their formulae (46)–(48) may be written as

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \mathscr{F} + O(\epsilon^2), \tag{48}$$

where $\mathscr{F}$ may be identified with $-\chi$. Brown & Mangler have checked that $\mathbf{u}_0$ may be identified with $\mathbf{u}_0^*$: the Brown (1965) solution. It is easily checked that, to the order considered,

$$\mathbf{u}_1 = -\frac{\nabla\chi_0 \cdot (\nabla \wedge \mathbf{u}_0^*)}{|\nabla\chi_0|^2} \tag{49}$$

and the agreement is complete.

5. Conclusion

The main result of this paper is that once the flow contains a rolled sheet with closely spaced turns one may avoid computing with the rolled sheet within the core. Provided we know an asymptotic representation of the core at some initial time, consistent with the double scale structure, then we are able to compute the evolution of this asymptotic representation. We need only compute the evolution of a continuous flow in the core region and transport the rolled sheet together with the corresponding discontinuities of the velocity and the entropy along the trajectories of the continuous flow. All this may seem to be rather transparent from a physical point of view and the main interest of the paper stands in the formal proof that this is consistent with a systematic scheme of expansion with respect to a small closeness parameter. In the same way one might take into account the effect of a small viscosity as in Guiraud & Zeytounian (1979*a*) but the present purpose was to derive the rule of equivalence in order to use it in numerical computations of inviscid flow, so that we do not embark on such a work. We think that it should be considered in the more general setting of solutions of Navier–Stokes equations having a multiple scale structure, but this would carry us far ahead of the limited scope of the present investigation.

A more fundamental question, raised by one referee, concerns the stability of these rolled sheets. A simple-minded idea would be to examine the behaviour against Kelvin–Helmholtz instability of the sheet as if it were planar. According to this, the sheet would be destroyed in a time $O(\lambda/C|\mathbf{u}_0^*|)$, and for wavelength of the order of the space between turns this time is $O(D/|\mathbf{u}_0^*|)$, where D is the diameter of the core, that is of the order of the time needed for a particle to make a few turns along the spiral. Of course this simple-minded argument may be completely irrelevant. As a matter of fact Moore (1976) following Moore & Griffith-Jones (1974) has shown that the stability problem may not be a local one. Anyway, Guiraud & Zeytounian (1979*b*) have presented an analysis which, again, shows that the stability problem is, in principle, not a local one. However for wavelengths fairly smaller than the spacing the mechanism responsible for instability is then the one for an infinite saw-tooth profile and it confirms that the growth time is $O(\lambda/C|\mathbf{u}_0^*|)$ with λ fairly smaller than $D/|\mathbf{u}_0^*|$. On the other hand, for $\lambda = O(D/|\mathbf{u}_0^*|)$ the stability analysis must involve the whole rolled vortex sheet in agreement with Moore (1976) conclusion. Anyway there is ample place for further analysis of this stability issue.

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